Optimal Maintenance of Systems with Markovian Mission and Deterioration

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Abstract

We consider the maintenance of a mission-based system that is designed to perform missions consisting of a random sequence of phases or stages with random durations. A finite state Markov process describes the mission process. The age or deterioration process of the system is described by another finite state Markov process whose generator depends on the phases of the mission. We discuss optimal replacement and optimal repair problems and characterize the optimal policies under some monotonicity assumptions. We also provide numerical illustrations to demonstrate the structure of the optimal policies.

Key words:
Optimal maintenance, mission-based system, Markovian deterioration

1. Introduction

Maintenance actions are vital for companies to increase reliability and availability of the production system and to decrease production costs. A significant portion of the total work force of a company is employed in maintenance departments and a great amount of money is spent for maintenance by companies every year. Therefore, optimal maintenance problems which address the obvious trade-off between maintenance costs and productivity are very important for both researchers and company managers. In this study, we analyze optimal replacement and repair problems for systems which are designed to perform missions consisting of different stages or phases. Such systems are called mission-based systems or phased-mission systems in the literature. The sequence and the duration of the phases can be deterministic or random, and all failure properties of the components as well as the configuration of the system change dramatically from phase to phase. These kind of systems were first introduced as phased-mission systems by Esary and Ziehms (1975) and a vast literature has accumulated since then. There are various models that involve systems with repairable and non-repairable components with deterministic or random phase durations and sequences. We refer the reader to Veatch (1986), Kim and Park (1994), Mura and Bondavalli (2001) and references cited in these papers for different models.

We assume that the mission process is a Markov process. In other words, the system under consideration performs a mission whose successive phases follow a Markov chain and all phase durations are exponentially distributed. We also assume that the system is subject to Markovian...
aging or deterioration. In other words, the successive deterioration levels of the system follow a Markov chain and holding times in each deterioration level are exponentially distributed during any phase. The most important point is that the generator of the deterioration process of the system (the transition probability matrix and rates of the holding times) depends on the mission process. This implies that the deterioration process is a Markov process modulated by another Markov process (i.e., the mission process). We characterize optimal replacement and repair policies which minimize the expected total discounted cost by considering phase dependent maintenance costs and state occupancy costs.

Optimal maintenance of systems subject to Markovian deterioration has been studied extensively in OR literature. One of the earliest and basic models where the deterioration process is described by a Markov chain is analyzed by Derman (1970). It is shown that the optimal policies minimizing both expected total discounted cost and expected average cost are control-limit type under some monotonicity assumptions on the costs and the deterioration process. In particular, there exists a critical deterioration level above which the optimal decision is “replace” and below which the optimal decision is “do nothing”. The same model with state occupancy costs incurred each time that the system is inspected is analyzed by Kolesar (1966) and the optimality of a control-limit policy is proved. A similar model with state dependent replacement costs is analyzed in Kawai et al. (2002). A generalization of the model in Kolesar (1966) is analyzed in Wood (1988) by considering the case where the replacement action may fail with some probability and the occupancy costs are not paid during replacement. The analysis is done by applying uniformization techniques by which a continuous-time Markov decision process can be converted into an equivalent discrete-time Markov decision process. The optimal policy is control-limit type provided that model parameters satisfy some monotonicity conditions.

Özekici and Günlük (1992) provide some sufficient conditions which make the lifetime of a system with Markovian deterioration increasing failure rate on average (IFRA), and also show that these conditions imply the optimality of a control-limit policy if the replacement cost does not depend on the deterioration level of the system. They also analyze optimal repair problems considering several different cost structures. In a related work, Özekici (1995) considers the maintenance problem of a device operating in a random environment and provides characterizations of optimal maintenance policies when the device ages intrinsically. In most of the literature, it is generally assumed that the system is working under a fixed environment or phase. Therefore, this study extends this line of research by examining systems with Markovian deterioration modulated by an external mission process. For more information on optimal maintenance problems, the interested readers are referred to the surveys by Jardine and Buzacott (1985), Cho and Parlar (1991), Wang (2002), and Nicolai and Dekker (2007).

In Section 2, we describe the stochastic structure of the mission and deterioration processes in detail. We analyze the optimal replacement problem in Section 3. In Section 4, the optimal repair problem is discussed and some special repair cost structures are further analyzed in Section 5. A number of illustrations are also provided in the Appendix.

2. Mission and Deterioration Processes

Let $X_t$ be the phase of the mission which is performed at time $t$. We assume that the mission process $X = \{X_t; t \geq 0\}$ is a Markov process with a finite state space $E$, transition probability matrix $Q$, and transition rate vector $\mu$. We also suppose that the deterioration level or age of the system takes values in some finite set $F = \{0, 1, \cdots, M\}$ where 0 stands for a
brand new system and $M$ represents system failure. The deterioration process of the system is represented by $A = \{A_t; t \geq 0\}$ with state space $F$. Since the survival properties of the system change depending on the phases of the mission process, we assume that $A$ is modulated by $X$. The deterioration process follows a Markov process with state space $F$, generator $G_i$, transition probability matrix $P_i$, and transition rate vector $\lambda_i$ during phase $i$.

We assume that all maintenance actions are instantaneous. We also suppose that $P_i$ is a stochastically monotone upper triangular matrix and failure rate of the system increases as the deterioration level of the system increases. In other words,

$$\sum_{b=k}^{M} P_i(a,b) \geq \sum_{b=k}^{M} P_i(a-1,b) \quad \text{and}$$

and

$$\lambda_i(a) \geq \lambda_i(a-1)$$

for all $i \in E, a = 1, \cdots, M$ and $k \in F$.

Considering the dependence between the deterioration process and the mission process, we will use the bivariate process $(X, A) = \{(X_t, A_t); t \geq 0\}$ which is more suitable for our purpose in the foregoing analysis. It is clear that $(X, A)$ is also a Markov process with state space $E \times F$ and generator matrix

$$G(i,a;j,b) = \begin{cases} G_i(a,b) & \text{if } a \neq M, j = i, b \neq a \\ H(i,j) & \text{if } a \neq M, j \neq i, b = a \\ G_i(a,a) + H(i,i) & \text{if } a \neq M, j = i, b = a \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j \in E$ and $a, b \in F$. Using $G$, the transition probability matrix of the imbedded Markov chain and the transition rates of $(X, A)$ can be obtained, respectively, as

$$P(i,a;j,b) = \begin{cases} \left( \frac{\lambda_i(a)}{\lambda_i(a)+\mu_i} \right) P_i(a,b) & \text{if } a \neq M, j = i, b \neq a \\ \left( \frac{\mu_i}{\lambda_i(a)+\mu_i} \right) Q(i,j) & \text{if } a \neq M, j \neq i, b = a \\ 1 & \text{if } b = a = M, j = i \\ 0 & \text{otherwise} \end{cases}$$

and

$$\lambda(i,a) = \begin{cases} \lambda_i(a) + \mu_i & \text{if } a \neq M \\ 0 & \text{if } a = M. \end{cases}$$

The structure of the stochastic model described in this section extends that of Özekici and Günlük (1992) in which there is Markovian deterioration and exponential failures. In our setting, we also include a Markovian mission-process that the system is designed to perform. Naturally, the deterioration depends on the phases of the mission and we now have a Markov modulated Markov process to maintain.

### 3. Optimal Replacement Problem

We will characterize the optimal replacement policy which minimizes the expected total discounted cost under the assumption that the decision maker is allowed to make a decision only
when a change occurs in the mission process or the deterioration process. There are three costs
associated with our problem. The preventive replacement and the failure costs during phase $i$
are $p_i$ and $f_i$ respectively with $f_i \geq p_i$ and $\sup_{i \in E} f_i = \bar{f} < +\infty$. A state occupancy cost $c(i, a)$
with $\sup_{i, a} c(i, a) = \bar{c} < +\infty$ is incurred if the system starts to perform phase $i$ with the initial
deterioration level $a$. It is assumed that $c(i, a)$ is increasing in $a$ for all $i \in E$. There is continuous
discounting some with rate $\alpha > 0$. Although we assume that $c(i, a)$ is a fixed lump-sum cost
incurred at the beginning of each decision epoch, it may also be incurred continuously. If the
state occupancy cost rate is $c_r(i, a)$ for the system performing phase $i$ with deterioration level
$a$, then it suffices to take

$$c(i, a) = \int_0^{+\infty} \lambda(i, a) e^{-\lambda(i, a)t} \int_0^t c_r(i, a)e^{-\alpha s}dsdt = \frac{c_r(i, a)}{\lambda(i, a) + \alpha}.$$  

We need to solve the dynamic programming equation (DPE)

$$v(i, a) = \min_{a \in A_s} \{r_s(i, a) + \Gamma_s v(i, a)\} \quad (4)$$

where $A_0 = \{0\}$, $A_M = \{1\}$, $A_s = \{0, 1\}$, $r_0(i, a) = c(i, a)$, $r_1(i, a) = c(i, 0) + p_i$, $r_1(i, M) =
c(i, 0) + f_i$

$$\Gamma_0 g(i, a) = \frac{\mu_i + \lambda_i(a)}{\mu_i + \lambda_i(0) + \alpha} \left[ \frac{\mu_i}{\mu_i + \lambda_i(0)} \sum_{j \in E} Q(i, j) g(j, a) + \frac{\lambda_i(0)}{\mu_i + \lambda_i(0)} \sum_{b=0}^M P_1(a, b) g(i, b) \right]$$

for $a = 0, \cdots, M - 1$, and

$$\Gamma_1 g(i, a) = \frac{\mu_i + \lambda_i(0)}{\mu_i + \lambda_i(0) + \alpha} \left[ \frac{\mu_i}{\mu_i + \lambda_i(0)} \sum_{j \in E} Q(i, j) g(j, 0) + \frac{\lambda_i(0)}{\mu_i + \lambda_i(0)} \sum_{b=0}^M P_1(0, b) g(i, b) \right] \quad (5)$$

for $a = 1, \cdots, M$.

Now, we can use a uniformization technique by applying the procedure in Puterman (1994). The
original model has

$$P_s(i, a; j, b) = \begin{cases} P(i, a; j, b) & \text{if } s = 0 \\
P(i, 0; j, b) & \text{if } s = 1 \end{cases}$$

and

$$\lambda_s(i, a) = \begin{cases} \lambda(i, a) & \text{if } s = 0 \\
\lambda(i, 0) & \text{if } s = 1 \end{cases}$$

where the subscripts of $P$ and $\lambda$ identifies the valid formula if the replacement decision denoted
by the subscript is applied.

To apply uniformization, we suppose that $\sup_{i \in E, a \in F} \{\mu_i + \lambda_i(a)\} = \bar{\lambda} < +\infty$ throughout
this paper. We define

$$\tilde{r}_s(i, a) = r_s(i, a) \left( \frac{\alpha + \lambda_s(i, a)}{\alpha + \bar{\lambda}} \right)$$

and

$$\tilde{P}_s(i, a; j, b) = \begin{cases} 1 - \frac{(1-P_s(i,a;j,b))\lambda_s(i,a)}{P_s(i,a;j,b)\lambda_s(i,a)} & \text{if } (j, b) = (i, a) \\
\frac{1-P_s(i,a;j,b)}{\lambda_s(i,a)} & \text{if } (j, b) \neq (i, a). \end{cases}$$
Then, by Proposition 11.5.1. in Puterman (1994), we have \( v(i, a) = \tilde{v}(i, a) \) for every stationary policy where \( \tilde{v} \) satisfies the DPE

\[
\tilde{v}(i, a) = \min_{s \in A_a} \left\{ \tilde{r}_s(i, a) + \tilde{\Gamma}_s \tilde{v}(i, a) \right\}
\]

for all \( i \in E \) and \( a \in F \) where

\[
\begin{align*}
\tilde{r}_0(i, a) &= c(i, a) \left( \frac{\alpha + \mu_i + \lambda_i(a)}{\alpha + \lambda} \right) \\
\tilde{r}_1(i, a) &= (c(i, 0) + p_i) \left( \frac{\alpha + \lambda_i(0) + \mu_i}{\alpha + \lambda} \right) \\
\tilde{r}_1(i, M) &= (c(i, 0) + f_i) \left( \frac{\alpha + \lambda_i(0) + \mu_i}{\alpha + \lambda} \right)
\end{align*}
\]

for all \( a \in F \setminus \{M\} \),

\[
\begin{align*}
\tilde{\Gamma}_0 g(i, a) &= \frac{\lambda}{\lambda + \alpha} \left[ \frac{\mu_i}{\lambda} \sum_{j \in E} Q(i, j) g(j, a) + \frac{\lambda_i(a)}{\lambda} \sum_{b=0}^{M} P_i(a, b) g(i, b) + \left( 1 - \frac{\mu_i + \lambda_i(a)}{\lambda} \right) g(i, a) \right] \\
\tilde{\Gamma}_1 g(i, a) &= \frac{\lambda}{\lambda + \alpha} \left[ \frac{\mu_i}{\lambda} \sum_{j \in E} Q(i, j) g(j, 0) + \frac{\lambda_i(0)}{\lambda} \sum_{b=0}^{M} P_i(0, b) g(i, b) + \left( 1 - \frac{\mu_i + \lambda_i(0)}{\lambda} \right) g(i, a) \right].
\end{align*}
\]

Let \( \mathfrak{B} \) denote the set of all real-valued bounded functions defined on \( E \times F \). For any \( f \in \mathfrak{B} \), we define the operator \( \Upsilon \) so that

\[
\Upsilon f(i, a) = \min_{s \in A_a} \left\{ \tilde{r}_s(i, a) + \tilde{\Gamma}_s f(i, a) \right\}
\]

for all \( i \in E \), and \( a \in F \).

**Lemma 1.** If \( g(i, a) = f(i, a) + h \) for some constant \( h \), then \( \Upsilon g(i, a) = \Upsilon f(i, a) + \left( \frac{\lambda}{\lambda + \alpha} \right) h \).

**Proof.** Using (7) and (8),

\[
\begin{align*}
\tilde{\Gamma}_0 g(i, a) &= \frac{\mu_i}{\lambda + \alpha} \sum_{j \in E} Q(i, j) g(j, a) + \frac{\lambda_i(a)}{\lambda + \alpha} \sum_{b=0}^{M} P_i(a, b) g(i, b) + \left( 1 - \frac{\mu_i + \lambda_i(a)}{\lambda + \alpha} \right) g(i, a) \\
&= \Gamma_0 f(i, a) + \frac{c}{c + \alpha} h
\end{align*}
\]

and

\[
\begin{align*}
\tilde{\Gamma}_1 g(i, a) &= \frac{\mu_i}{\lambda + \alpha} \sum_{j \in E} Q(i, j) g(j, 0) + \frac{\lambda_i(0)}{\lambda + \alpha} \sum_{b=0}^{M} P_i(0, b) g(i, b) + \left( 1 - \frac{\mu_i + \lambda_i(0)}{\lambda + \alpha} \right) g(i, a) \\
&= \Gamma_1 f(i, a) + \frac{\lambda}{\lambda + \alpha} h.
\end{align*}
\]
This implies that
\[
\Upsilon g (i, a) = \min_{s \in A_a} \left\{ \tilde{r}_s (i, a) + \tilde{\Gamma}_s f (i, a) + \left( \frac{\bar{\lambda}}{\lambda + \alpha} \right) h \right\} = \Upsilon f (i, a) + \left( \frac{\bar{\lambda}}{\lambda + \alpha} \right) h.
\]

\[\square\]

**Theorem 1.** There is a unique function in \(\mathcal{B}\) which satisfies the DPE (6).

**Proof.** We will use Banach contraction mapping theorem. Choose two functions \(f, g \in \mathcal{B}\) and suppose that \(\|\cdot\|\) is the usual supremum norm. Let \(h = \|f - g\|\). Then, we can write
\[
g (i, a) - h \leq f (i, a) \leq g (i, a) + h
\]
for all \(i \in E\) and \(a \in F\). It is easy to see that
\[
\Upsilon (g (i, a) - h) \leq \Upsilon f (i, a) \leq \Upsilon (g (i, a) + h).
\]
Then, using Lemma 1,
\[
\Upsilon g (i, a) - h \left( \frac{\bar{\lambda}}{\lambda + \alpha} \right) \leq \Upsilon f (i, a) \leq \Upsilon g (i, a) + h \left( \frac{\bar{\lambda}}{\lambda + \alpha} \right)
\]
and this further implies that
\[
|\Upsilon f (i, a) - \Upsilon g (i, a)| \leq h \left( \frac{\bar{\lambda}}{\lambda + \alpha} \right)
\]
for all \(i \in E\) and \(a \in F\) so that
\[
\|\Upsilon f - \Upsilon g\| \leq \left( \frac{\bar{\lambda}}{\lambda + \alpha} \right) \|f - g\|.
\]
This implies that \(\Upsilon\) is a contraction mapping since \(\bar{\lambda}/ (\lambda + \alpha) < 1\), and it has a unique fixed point in \(\mathcal{B}\) which satisfies the DPE (6) by Banach’s contraction mapping theorem. \[\square\]

A consequence of Theorem 1 is that there is a stationary policy which solves the DPE (6). Since, \(\bar{v} = v\) for every stationary policy, they are equal for the optimal replacement policy. Therefore, if we have any characterization for the transformed process, the same characterization will be valid for the original process. From now on, \(v\) and \(\bar{v}\) will represent the expected total discounted cost under the optimal replacement policy for the original process and the transformed process respectively. Our aim is to characterize the optimal policy. This is accomplished by a sequence of results that follow.

**Lemma 2.** If \(\bar{v} (i, a)\) is increasing in \(a\), then
\[
\frac{\lambda_i (a)}{\bar{\lambda}} \sum_{b \geq a+1} P_i (a, b) \bar{v} (i, b) + \left( 1 - \frac{\mu_i + \lambda_i (a)}{\bar{\lambda}} \right) \bar{v} (i, a)
\]
is also increasing in \(a\) for all \(i \in E\).
Proof. Define two vectors as

\[ u_1 = \begin{bmatrix} \frac{\lambda-(\lambda_i+\mu_i)}{\lambda} & \frac{\lambda_i(a)}{\lambda} & \frac{\lambda(a)}{\lambda} & \ldots & \frac{\lambda_i(a)}{\lambda} & \frac{\lambda(a)}{\lambda} \end{bmatrix} \]

and

\[ u_2 = \begin{bmatrix} 0 & \frac{\lambda-(\lambda_i(a+1)+\mu_i)}{\lambda} & \frac{\lambda_i(a+1)}{\lambda} & \frac{\lambda(a+1)}{\lambda} & \ldots & \frac{\lambda_i(a+1)}{\lambda} & \frac{\lambda(a+1)}{\lambda} \end{bmatrix}. \]

We first show that

\[ \sum_{i=k}^{M-a+1} u_2(i) \geq \sum_{i=k}^{M-a+1} u_1(i) \]

for all \( k = 1, \ldots, M - a + 1. \) If \( k = 1, \)

\[ \sum_{i=1}^{M-a+1} u_1(i) = \frac{\lambda - \mu_i}{\lambda} = \sum_{i=1}^{M-a+1} u_2(i) = \frac{\lambda - \mu_i}{\lambda}. \]

If \( k = 2, \)

\[ \sum_{i=2}^{M-a+1} u_1(i) = \frac{\lambda_i(a)}{\lambda} \leq \sum_{i=2}^{M-a+1} u_2(i) = \frac{\hat{\lambda} - \mu_i}{\lambda}. \]

since \( \lambda_i(0) + \mu_i \leq \hat{\lambda}. \)

If \( k = j \) for some \( j = 3, \ldots, M - a + 1, \) then using (1) and (2)

\[ \sum_{i=j}^{M-a+1} u_1(i) = \frac{\lambda_i(a)}{\lambda} \sum_{b=a+j-1}^{M-a+1} P_i(a, b) \leq \sum_{i=j}^{M-a+1} u_2(i) = \frac{\lambda_i(a+1)}{\lambda} \sum_{b=a+j-1}^{M-a+1} P_i(a+1, b). \]

Thus, we have

\[ \sum_{i=k}^{M-a+1} u_2(i) \geq \sum_{i=k}^{M-a+1} u_1(i) \]

for all \( k = 1, \ldots, M - a + 1. \) Moreover, since \( \tilde{v} \) is increasing in \( a, \) using Lemma 1 on page 123 in Derman (1970), we have

\[ \sum_{j=a}^{M} u_2(j-a+1) \tilde{v}(i,j) \geq \sum_{j=a}^{M} u_1(j-a+1) \tilde{v}(i,j) \]

and this completes the proof. \( \square \)

**Theorem 2.** \( \tilde{v}(i, a) \) is increasing in \( a \) and bounded by \( (\tilde{\lambda} + \alpha)(\tilde{c} + \tilde{f})/\alpha. \)

**Proof.** It is sufficient to show that \( \tilde{r}_s(i, a) + \tilde{\gamma}_s \tilde{v}(i, a) \) is increasing in \( a \) and bounded by \( (\tilde{\lambda} + \alpha)(\tilde{c} + \tilde{f})/\alpha \) from above for each value of \( s \) assuming that \( \tilde{v}(i, a) \) is increasing in \( a \) and bounded by \( (\tilde{\lambda} + \alpha)(\tilde{c} + \tilde{f})/\alpha \) from above. It is trivial that

\[ \tilde{r}_s(i, a) \leq \tilde{c} + \tilde{f} \]
and 
\[ \tilde{\Gamma}_s v(i, a) \leq \left( \frac{\bar{\lambda}}{\lambda + \alpha} \right) \frac{(\bar{\lambda} + \alpha)(\bar{\tau} + \bar{f})}{\alpha} \]
for each value of \( s \). These imply that 
\[ \tilde{\tau}_s(i, a) + \tilde{\Gamma}_s v(i, a) \leq (\bar{\lambda} + \alpha)(\bar{\tau} + \bar{f})/\alpha. \]

Choose arbitrary \( a \in F \setminus \{M\} \) and first assume that \( s = 0 \). Since, \( \tilde{\tau}_0(i, a + 1) \geq \tilde{\tau}_0(i, a) \), it is sufficient to show that \( \tilde{\Gamma}_0 v(i, a + 1) \geq \tilde{\Gamma}_0 v(i, a) \). We have \( \tilde{v}(j, a + 1) \geq \tilde{v}(j, a) \) for all \( a \) by the main hypothesis and using Lemma 2, we have the desired result. Now suppose that \( s = 1 \). It is trivial that \( \tilde{\Gamma}_1 v(i, a + 1) \geq \tilde{\Gamma}_1 v(i, a) \) by the main hypothesis and \( \tilde{\tau}_1(i, a + 1) \geq \tilde{\tau}_1(i, a) \). This implies that \( \tilde{\tau}_1(i, a) + \tilde{\Gamma}_1 v(i, a) \) is increasing in \( a \) and this completes the proof. \[ \square \]

Immediate consequences of Theorem 2 yield the following characterizations.

**Corollary 1.** \( v(i, a) \) is increasing in \( a \) and bounded by \((\bar{\lambda} + \alpha)(\bar{\tau} + \bar{f})/\alpha. \)

**Proof.** This simply follows from Proposition 11.5.1 in Puterman (1994). \[ \square \]

**Corollary 2.** Suppose that \( s^* \) is the optimal replacement policy. Then, there exists \( a_i^* \) for every \( i \in E \) such that 
\[ s^*(i, a) = \begin{cases} 1 & \text{if } a \geq a_i^* \\ 0 & \text{if } a < a_i^* \end{cases}. \]

**Proof.** Choose arbitrary \( i \in E \). It suffices to show that if \( s^*(i, a) = 1 \), then \( s^*(i, b) = 1 \) for all \( b \geq a \) since \( s^*(i, M) = 1 \). Assume that \( s^*(i, a) = 1 \) for some \( a < M \) and, for a contradiction, suppose that there exists \( b > a \) such that \( s^*(i, b) = 0 \). Note that if we can not find such an \( a \), there is nothing to prove. Since, \( s^*(i, a) = 1 \), we have \( r_1(i, a) + \Gamma_1 v(i, a) \leq r_0(i, a) + \Gamma_0 v(i, a) \) and \( v(i, a) = r_1(i, a) + \Gamma_1 v(i, a) \). Since \( s^*(i, b) = 0 \), it follows that \( v(i, b) = r_0(i, b) + \Gamma_0 v(i, b) < r_1(i, b) + \Gamma_1 v(i, b) = r_1(i, a) + \Gamma_1 v(i, a) = v(i, a) \). But, this result is a contradiction by Corollary 1 and the proof is completed. \[ \square \]

Thus, we proved that the optimal replacement policy is a phase-dependent control-limit policy on the deterioration level of the system. The critical replacement levels depend on the phases of the mission. The optimal replacement policy must be as depicted in Figure 1 for a system performing a mission with two phases. However, this simple structure may not be optimal if our assumptions are violated, as shown by some examples in the Appendix.

### 4. Optimal Repair Problem

In the previous section, we analyzed the optimal replacement problem in which a decision maker observes the system at the beginning of each decision epoch and then makes a decision on replacing the system or not. However, repairing the system to a better state is also possible in real-life applications in addition to the replacement option. In this section, we assume that after any change in the deterioration process or in the mission process, a decision maker observes the system and then decides to repair the system to a better state immediately or to do nothing.

Let \( C_i(a, b) \) be the cost of repairing the system from deterioration level \( a \) to deterioration level \( b \) during phase \( i \). It is assumed that \( C_i(a, b) \) is increasing in \( a \) for a fixed \( b \), decreasing in \( b \) for a given \( a \), and \( C_i(a, a) = 0 \). It is also assumed that for any initial deterioration level
If \( a < M \), we can repair the system to any deterioration level in the set \( \{0, 1, \ldots, a\} \) with the option of doing nothing. However, a failed component must be replaced. In other words, a failed component (with deterioration level \( M \)) can only be repaired to the deterioration level 0. We let \( v(i, a) \) denote the expected total discounted cost given that the initial phase is \( i \in E \) and the initial deterioration level of the system is \( a \in F \).

We need to solve the DPE

\[
v(i, a) = \min_{b \in A(a)} \left\{ C_i(a, b) + c(i, b) + \Gamma v(i, b) \right\}
\]

where

\[
\Gamma v(i, a) = \frac{\mu_i + \lambda_i(a)}{\mu_i + \lambda_i(a)} \sum_{j \in E} Q(i, j) v(j, a) + \frac{\lambda_i(a)}{\mu_i + \lambda_i(a)} \sum_{b=0}^{M} P_i(a, b) v(i, b)
\]

and

\[
A(a) = \begin{cases} 
\{0, 1, \ldots, a\} & \text{if } a < M \\
\{0\} & \text{if } a = M.
\end{cases}
\]

It is assumed that in the existence of a tie, the decision maker chooses the smaller deterioration level to which the system will be repaired. We further suppose that \( \sup_{i \in E} C_i(M, 0) = \overline{C} < +\infty \).

**Theorem 3.** There is a unique function in \( \mathfrak{B} \) that satisfies the DPE (11).

**Proof.** Define the operator \( \Upsilon \) so that

\[
\Upsilon f(i, a) = \min_{b \in A(a)} \left\{ C_i(a, b) + c(i, b) + \Gamma f(i, b) \right\}
\]

for any \( f \in \mathfrak{B} \). We will use Banach’s contraction mapping theorem. Choose two functions \( g \) and \( h \) from \( \mathfrak{B} \) and assume that \( \| \cdot \| \) is the usual supremum norm. Consider

\[
\Upsilon g(i, a) - \Upsilon h(i, a) = \min_{b \in A(a)} \left\{ C_i(a, b) + c(i, b) + \Gamma g(i, b) \right\} - \min_{b \in A(a)} \left\{ C_i(a, b) + c(i, b) + \Gamma h(i, b) \right\}.
\]
Suppose that \( \bar{b} \in A(a) \) minimizes the second term in the right hand side of (13). Then,

\[
\begin{align*}
\mathcal{Y} g(i, a) - \mathcal{Y} h(i, a) & \leq C_i(a, \bar{b}) + c(i, \bar{b}) + \Gamma g(i, \bar{b}) - C_i(a, \bar{b}) - c(i, \bar{b}) - \Gamma h(i, \bar{b}) \\
& \leq \frac{\mu_i + \lambda_i(\bar{b})}{\mu_i + \lambda_i(\bar{b}) + \alpha} \left[ \frac{\mu_i}{\mu_i + \lambda_i(\bar{b})} \sum_{j \in E} Q(i, j) (g(j, \bar{b}) - h(j, \bar{b})) \right] \\
& \quad + \frac{\lambda_i(\bar{b})}{\mu_i + \lambda_i(\bar{b})} \left[ \sum_{b=0}^{M} P_i(\bar{b}, b) (g(i, b) - h(\bar{b}, b)) \right] \\
& \leq \sup_{i \in E, a \in F} \left\{ \frac{\mu_i + \lambda_i(a)}{\mu_i + \lambda_i(a) + \alpha} \right\} \| g - h \|
\end{align*}
\]

Similarly, it can be shown that

\[
\mathcal{Y} h(i, a) - \mathcal{Y} g(i, a) \leq \sup_{i \in E, a \in F} \left\{ \frac{\mu_i + \lambda_i(a)}{\mu_i + \lambda_i(a) + \alpha} \right\} \| g - h \|
\]

Thus, we have

\[
\| \mathcal{Y} g - \mathcal{Y} h \| \leq \sup_{i \in E, a \in F} \left\{ \frac{\mu_i + \lambda_i(a)}{\mu_i + \lambda_i(a) + \alpha} \right\} \| g - h \|
\]

Then,

\[
\sup_{i \in E, a \in F} \left\{ \frac{\mu_i + \lambda_i(a)}{\mu_i + \lambda_i(a) + \alpha} \right\} = \sup_{i \in E, a \in F} \left\{ \frac{1 - \frac{\alpha}{\mu_i + \lambda_i(a)}}{\frac{\mu_i + \lambda_i(a)}{\mu_i + \lambda_i(a) + \alpha}} \right\} = \frac{1}{\lambda + \alpha} < 1.
\]

This implies that \( \mathcal{Y} \) is a contraction mapping and it has a unique fixed point in \( \mathcal{B} \) that satisfies the DPE (11). \( \square \)

Using a similar uniformization technique, the DPE (11) can be rewritten as

\[
v(i, a) = \min_{s \in A(a)} \{ r_s(i, a) + \Gamma_s v(i, a) \} \tag{14}
\]

where \( r_s(i, a) = C_i(a, s) + c(i, s) \), and

\[
\Gamma_s v(i, a) = \frac{\mu_i + \lambda_i(s)}{\mu_i + \lambda_i(s) + \alpha} \left[ \frac{\mu_i}{\mu_i + \lambda_i(s)} \sum_{j \in E} Q(i, j) v(j, s) + \frac{\lambda_i(s)}{\mu_i + \lambda_i(s)} \sum_{b=s+1}^{M} P_i(s, b) v(i, b) \right].
\]

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The original model has \( \lambda_s(i, a) = \mu_i + \lambda_i(s) \) and

\[
P_s(i, a; j, b) = \begin{cases} \frac{\mu_i}{\mu_i + \lambda_i(s)} Q(i, j) & \text{if } b = s, j \neq i \\ \frac{\lambda_i(s)}{\mu_i + \lambda_i(s)} P_i(s, b) & \text{if } b \neq s, j = i. \end{cases}
\]

where the subscripts of \( P \) and \( \lambda \) identify the valid formula if the repair decision denoted by the subscript is applied. Note that \( \lambda_s(i, a) \leq \bar{\lambda} \) for all \( i \in E, a \in F \), and \( s \in A(a) \), which is a necessary condition for the uniformization technique.

Now, we define

\[
\tilde{r}_s(i, a) = r_s(i, a) \left( \frac{\alpha + \lambda_s(i, a)}{\alpha + \bar{\lambda}} \right)
\]

and

\[
\tilde{P}_s(i, a; j, b) = \begin{cases} 1 - \frac{(1 - P_s(i, a; j, b)) \lambda_s(i, a)}{\bar{\lambda}} & \text{if } (j, b) = (i, a) \\ \frac{\lambda_s(i, a)}{\bar{\lambda} P_s(i, a; j, b)} & \text{if } (j, b) \neq (i, a). \end{cases}
\]

Then, by Proposition 11.5.1 in Puterman (1994), \( v(i, a) = \tilde{v}(i, a) \) for every stationary policy where \( \tilde{v} \) satisfies the new DPE

\[
\tilde{v}(i, a) = \min_{s \in A(a)} \left\{ \tilde{r}_s(i, a) + \tilde{P}_s(i, a; E, a) \tilde{v}(i, a) \right\}
\]

for all \( i \in E \) and \( a \in F \), where

\[
\tilde{\Gamma}_a \tilde{v}(i, a) = \frac{\bar{\lambda}}{\alpha + \bar{\lambda}} \left[ \frac{\mu_i}{\lambda} \sum_{j \in E} Q(i, j) \tilde{v}(j, s) + \frac{\lambda_i(s)}{\lambda} \sum_{b = s+1}^{M} P_i(s, b) \tilde{v}(i, b) \right.
\]

\[
\left. + \left( 1 - \frac{\mu_i + \lambda_i(s)}{\lambda} \right) \tilde{v}(i, a) \right].
\]

We now follow the footsteps of the previous section and provide a number of characterizations of the optimal policy. There is an optimal stationary policy and \( \tilde{v} = v \) where these represent the expected total discounted cost under the optimal repair policy for the original process and the transformed process respectively. If we have any characterization for the transformed process, the same characterization will be valid for the original process as well.

**Lemma 3.** If \( \tilde{v}(i, a) \) is increasing in \( a \) for all \( i \in E \), then \( \tilde{\Gamma}_a \tilde{v}(i, a) \) is also increasing in \( a \) for all \( i \in E \).

**Proof.** Choose arbitrary \( a \in \{0, 1, \ldots, M - 2\} \). Then, we have

\[
\tilde{\Gamma}_{a+1} \tilde{v}(i, a + 1) = \frac{\bar{\lambda}}{\alpha + \bar{\lambda}} \left[ \frac{\mu_i}{\lambda} \sum_{j \in E} Q(i, j) \tilde{v}(j, a + 1) + \frac{\lambda_i(a + 1)}{\lambda} \sum_{b \geq a+2} P_i(a + 1, b) \tilde{v}(i, b) \right.
\]

\[
\left. + \left( 1 - \frac{\mu_i + \lambda_i(a + 1)}{\lambda} \right) \tilde{v}(i, a + 1) \right]
\]

and

\[
\tilde{\Gamma}_a \tilde{v}(i, a) = \frac{\bar{\lambda}}{\alpha + \bar{\lambda}} \left[ \frac{\mu_i}{\lambda} \sum_{j \in E} Q(i, j) \tilde{v}(j, a) + \frac{\lambda_i(a)}{\lambda} \sum_{b \geq a+1} P_i(a, b) \tilde{v}(i, b) \right.
\]

\[
\left. + \left( 1 - \frac{\mu_i + \lambda_i(a)}{\lambda} \right) \tilde{v}(i, a) \right].
\]
Since $\tilde{v}(j, a + 1) \geq \tilde{v}(j, a)$ for every $j$ and $\lambda_i (a + 1) \geq \lambda_i (a)$ we have

$$\frac{\lambda_i (a)}{\lambda} \sum_{b \geq a + 1} P^i (a, b) \tilde{v} (i, b) + \left( 1 - \frac{\mu_i + \lambda_i (a)}{\lambda} \right) \tilde{v} (i, a)$$

$$\leq \frac{\lambda_i (a + 1)}{\lambda} \sum_{b \geq a + 2} P^i (a + 1, b) \tilde{v} (i, b) + \left( 1 - \frac{\mu_i + \lambda_i (a + 1)}{\lambda} \right) \tilde{v} (i, a + 1)$$

using Lemma 2 and this completes the proof. \(\square\)

**Theorem 4.** $\tilde{v}(i, a)$ is increasing in $a$ and bounded by $(\bar{\lambda} + \alpha) (\bar{C} + \bar{v}) / \alpha$.

**Proof.** Define the operator $\tilde{\Upsilon}$ so that

$$\tilde{\Upsilon} f (i, a) = \min_{s \in A (a)} \left\{ \tilde{r}_s (i, a) + \tilde{\Gamma}_s \tilde{v} (i, a) \right\}$$

for any $f \in \mathfrak{B}$. It suffices to show that if $\tilde{v} (i, a)$ is increasing in $a$ and bounded by $(\bar{\lambda} + \alpha) (\bar{C} + \bar{v}) / \alpha$, then $\tilde{\Upsilon} \tilde{v} (i, a)$ is increasing in $a$ and bounded by $(\bar{\lambda} + \alpha) (\bar{C} + \bar{v}) / \alpha$. It is clear that $\tilde{r}_s (i, a) \leq \bar{C} + \bar{v}$ and

$$\tilde{\Gamma}_s \tilde{v} (i, a) \leq \frac{\bar{\lambda}}{\bar{\lambda} + \alpha} \frac{(\bar{\lambda} + \alpha) (\bar{C} + \bar{v})}{\alpha} = \frac{\bar{\lambda} (\bar{C} + \bar{v})}{\alpha}.$$

These trivially imply that

$$\tilde{r}_s (i, a) + \tilde{\Gamma}_s \tilde{v} (i, a) \leq \frac{(\bar{\lambda} + \alpha) (\bar{C} + \bar{v})}{\alpha}$$

and this proves that $\tilde{\Upsilon} \tilde{v}$ is bounded by $(\bar{\lambda} + \alpha) (\bar{C} + \bar{v}) / \alpha$. Since $\tilde{r}_s (i, a)$ and $\tilde{v} (i, a)$ are increasing in $a$, using (15), we have

$$\tilde{r}_s (i, a + 1) + \tilde{\Gamma}_s \tilde{v} (i, a + 1) \geq \tilde{r}_s (i, a) + \tilde{\Gamma}_s \tilde{v} (i, a) \quad (16)$$

for any $s \in \{0, 1, \ldots , a\}$ if $a \in \{0, 1, \ldots , M - 2\}$ and for $s = 0$ if $a = M - 1$. Then, if

$$\tilde{\Upsilon} \tilde{v} (i, a + 1) = \tilde{r}_{s_{a+1}} (i, a + 1) + \tilde{\Gamma}_{s_{a+1}} \tilde{v} (i, a + 1)$$

for some $s_{a+1} \in \{0, 1, \ldots , a\}$, using (16),

$$\tilde{\Upsilon} \tilde{v} (i, a + 1) = \tilde{r}_{s_{a+1}} (i, a + 1) + \tilde{\Gamma}_{s_{a+1}} \tilde{v} (i, a + 1) \geq \tilde{r}_{s_{a+1}} (i, a) + \tilde{\Gamma}_{s_{a+1}} \tilde{v} (i, a) \geq \tilde{\Upsilon} \tilde{v} (i, a).$$

If

$$\tilde{\Upsilon} \tilde{v} (i, a + 1) = \tilde{r}_{a+1} (i, a + 1) + \tilde{\Gamma}_{a+1} \tilde{v} (i, a + 1)$$

where $a \in \{0, 1, \ldots , M - 2\}$ necessarily, then using Lemma 3,

$$\tilde{\Upsilon} \tilde{v} (i, a + 1) = \tilde{r}_{a+1} (i, a + 1) + \tilde{\Gamma}_{a+1} \tilde{v} (i, a + 1) \geq \tilde{r}_{a} (i, a) + \tilde{\Gamma}_{a} \tilde{v} (i, a) \geq \tilde{\Upsilon} \tilde{v} (i, a)$$

where the first inequality follows from

$$\tilde{r}_{a+1} (i, a + 1) = \frac{c (i, a + 1) (\alpha + \mu_i + \lambda_i (a + 1))}{\alpha + \lambda} \geq \frac{c (i, a) (\alpha + \mu_i + \lambda_i (a))}{\alpha + \lambda} = \tilde{r}_{a} (i, a)$$

since $c (i, a)$ and $\lambda_i (a)$ are increasing in $a$. Since $\tilde{\Upsilon} \tilde{v} (i, a + 1) \geq \tilde{\Upsilon} \tilde{v} (i, a)$ for any $a \in F \setminus \{M\}$ in all possible cases, the proof is completed. \(\square\)
An immediate corollary of this theorem is given below, which follows from Proposition 11.5.1 in Puterman (1994).

**Corollary 3.** \( v(i, a) \) is increasing in \( a \) for all \( i \in E \) and bounded by \( (\lambda + \alpha) \left( \overline{C} + c \right) / \alpha \).

From now on, we analyze the structure of the optimal repair policy. Let \( r_i(a) \) be the optimal repair decision during phase \( i \) if the deterioration level of the system is \( a \). We define the marginal repair cost \( \nabla C_i(a, b) \), for \( b \leq a \), as

\[
\nabla C_i(a, b) = C_i(a, b - 1) - C_i(a, b).
\]

Then, we have

\[
C_i(a, b) = \sum_{k=b+1}^{a} \nabla C_i(a, k) \tag{17}
\]

for all \( b < a \).

We will characterize the optimal repair policy by making some additional assumptions on the repair costs.

**Assumption 1.** For a given \( i \in E \), \( C_i(a, b) \leq C_i(a, k) + C_i(k, b) \), for all \( b, k, a \in F \) such that \( b \leq k \leq a \).

**Assumption 2.** For a given \( i \in E \), \( \nabla C_i(a, b) \) is increasing in \( a \) on \( \{k \in F; k \geq b\} \), for all fixed \( b \in F \setminus \{M\} \).

**Assumption 3.** For a given \( i \in E \), \( \nabla C_i(a, b) \) is increasing in \( a \) on \( \{k \in F; k > b\} \), for all fixed \( b \in F \setminus \{M\} \).

Assumption 1 states that the cost of repairing the system from deterioration level \( a \) to deterioration level \( b \) is less than or equal to the cost of applying two successive repair actions which take the deterioration level of the system first from \( a \) to an intermediate deterioration level \( k \) and then from \( k \) to \( b \). If there exists a fixed cost associated with each repair action, then this assumption is quite reasonable. Assumption 2 and Assumption 3 state that the marginal cost of repairing the system to a fixed state is increasing in the deterioration level of the system. This assumption is also quite reasonable, since in real life the cost of making the same amount of improvement in the state of a system generally increases as the deterioration level of the system increases. It is clear that Assumption 2 is stronger than Assumption 3, since its requires that the condition must hold when \( k = b \). Thus, besides what Assumption 3 states, Assumption 2 additionally states that \( \nabla C_i(a, a) \leq \nabla C_i(a + 1, a) \), or \( C_i(a + 1, a) + C_i(a, a - 1) \leq C_i(a + 1, a - 1) \).

Assumption 1 simply states that for the same amount of improvement, a direct repair is better than successive repairs. An immediate consequence of this assumption is the following theorem.

**Theorem 5.** If Assumption 1 holds, then \( r_i(r_i(a)) = r_i(a) \) for all \( a \in F \).

**Proof.** If \( r_i(a) = a \), then the result is trivial. Suppose that \( r_i(a) = b < a \) and \( r_i(b) = c < b \) for a contradiction. We have

\[
v(i, a) = C_i(a, b) + c(i, b) + \Gamma v(i, b) < C_i(a, c) + c(i, c) + \Gamma v(i, c) \tag{18}
\]
and
\[ v(i, b) = C_i(b, c) + c(i, c) + \Gamma v(i, c) \leq c(i, b) + \Gamma v(i, b). \tag{19} \]

Then, using (18),
\[ C_i(a, b) - C_i(a, c) < c(i, c) + \Gamma v(i, c) - c(i, b) - \Gamma v(i, b) \]
and using (19),
\[ \Gamma v(i, c) - \Gamma v(i, b) \leq c(i, b) - c(i, c) - C_i(b, c). \]
These imply that
\[ C_i(a, b) + C_i(b, c) < C_i(a, c). \]
This result clearly contradicts Assumption 1. \qed

**Theorem 6.** If Assumption 2 holds, then the optimal repair policy \( r_i \) is increasing on \( F \setminus \{M\} \) for all \( i \in E \).

**Proof.** For a contradiction, suppose that \( r_i(a_1) = b \) and \( r_i(a_2) = c < b \) where \( a_2 > a_1 \). Then, we have
\[
\begin{align*}
v(i, a_1) &= C_i(a_1, b) + c(i, b) + \Gamma v(i, b) < C_i(a_1, c) + c(i, c) + \Gamma v(i, c) \\
v(i, a_2) &= C_i(a_2, c) + c(i, c) + \Gamma v(i, c) \leq C_i(a_2, b) + c(i, b) + \Gamma v(i, b)
\end{align*}
\]
and these imply that
\[ C_i(a_2, c) - C_i(a_2, b) < C_i(a_1, c) - C_i(a_1, b). \]

Using (17),
\[
\sum_{j=c+1}^{a_2} \nabla C_i(a_2, j) - \sum_{j=b+1}^{a_2} \nabla C_i(a_2, j) < \sum_{j=c+1}^{a_1} \nabla C_i(a_1, j) - \sum_{j=b+1}^{a_1} \nabla C_i(a_1, j)
\]
and
\[ \sum_{j=c+1}^{b} \nabla C_i(a_2, j) < \sum_{j=c+1}^{b} \nabla C_i(a_1, j) \]
and, hence,
\[
\sum_{j=c+1}^{b} [\nabla C_i(a_2, j) - \nabla C_i(a_1, j)] < 0. \tag{20}
\]
This is a contradiction since every term in (20) is nonnegative by Assumption 2. \qed

**Theorem 7.** If Assumption 3 holds, then the optimal repair policy \( r_i \) is increasing on
\[ \{b \in F \setminus \{M\}; r_i(b) < b\} \]
for all \( i \in E \), i.e., if \( r_i(a) = b < a \), then \( r_i(c) \geq b \) for all \( c > a \).
Proof. For a contradiction, suppose that \( r_i(a_1) = b < a_1 \) and \( r_i(a_2) = c < b \) where \( a_2 > a_1 \). Then, we have

\[
v(i, a_1) = C_i(a_1, b) + \Gamma v(i, b) < C_i(a_1, c) + \Gamma v(i, c) \]
\[
v(i, a_2) = C_i(a_2, c) + \Gamma v(i, c) \leq C_i(a_2, b) + \Gamma v(i, b)
\]

and these imply that

\[ C_i(a_2, c) - C_i(a_2, b) < C_i(a_1, c) - C_i(a_1, b) \).

Using (17),

\[
\sum_{j = c+1}^{a_2} \nabla C_i(a_2, j) - \sum_{j = b+1}^{a_2} \nabla C_i(a_2, j) < \sum_{j = c+1}^{a_1} \nabla C_i(a_1, j) - \sum_{j = b+1}^{a_1} \nabla C_i(a_1, j)
\]

and

\[
\sum_{j = c+1}^{b} \nabla C_i(a_2, j) < \sum_{j = c+1}^{b} \nabla C_i(a_1, j)
\]

and, hence,

\[
\sum_{j = c+1}^{b} [\nabla C_i(a_2, j) - \nabla C_i(a_1, j)] < 0. \tag{21}
\]

This is a contradiction since every term in (20) is nonnegative by Assumption 3. \( \square \)

The main difference between Theorem 6 and Theorem 7 is the following. Theorem 6 holds when \( r_i(a_1) = a_1 \), but Theorem 7 may not hold in this case. If \( r_i(a_1) = a_1 \), following the same steps in the proof of Theorem 7 we can achieve the result

\[
\sum_{j = c+1}^{b} [\nabla C_i(a_2, j) - \nabla C_i(a_1, j)] < 0. \tag{22}
\]

Since \( a_1 = b \), \( \nabla C_i(a_2, b) - \nabla C_i(a_1, b) \) may be negative according to Assumption 3 and (22) may hold.

**Theorem 8.** If Assumption 1 and Assumption 2 hold, \( r_i(a) < a \) implies that \( r_i(a) = r_i(a-1) \) for all \( a \in F \setminus \{M\} \).

**Proof.** If \( r_i(a-1) = a-1 \), then \( r_i(a) = a-1 = r_i(a-1) \) trivially using Theorem 6. Now, suppose that \( r_i(a-1) = k < a-1 \). Choose arbitrary \( b \) such that \( k + 1 \leq b \leq a - 1 \). If \( r_i(a) = b \), then \( r_i(b) = b > k \) using Theorem 5. However, this contradicts Theorem 6 since \( b \leq a - 1 \) and \( r_i(b) > r_i(a-1) \). Thus, we have \( r_i(a) \notin \{k+1, k+2, \cdots, a-1\} \). Since \( k = r_i(a-1) \leq r_i(a) < a \) by Theorem 6, \( r_i(a) = k = r_i(a-1) \) and this completes the proof. \( \square \)

An immediate corollary of this theorem follows.

**Corollary 4.** If Assumption 1 and Assumption 2 hold, then \( r_i(a+1) \in \{r_i(a), a+1\} \) for \( a = 0, 1, \cdots, M - 2 \).

**Proof.** Suppose that \( r_i(a+1) < a+1 \). If \( r_i(a) = a \), then \( r_i(a+1) = r_i(a) = a \) since \( r_i(a+1) \geq r_i(a) \) by Theorem 6. If \( r_i(a) < a \), then \( r_i(a+1) = r_i(a) \) by Theorem 8. \( \square \)
Theorem 8 and Corollary 4 imply that the optimal repair policy for a given phase $i$ must be as depicted by Figure 2 under a cost structure for which Assumption 1 and Assumption 2 hold. Observe that the optimal policy has the form of an increasing step function. For levels $x, x+1,$ and $x+2,$ the optimal policy is to repair to level $x,$ for levels $y, y+1, y+2,$ and $y+3,$ the optimal policy is to repair to level $y.$ Also, note that the optimal policy is do nothing in $z, z+1,$ and $z+2.$ Illustrations are given in the appendix to show that this structure may not be true in case our assumptions are violated.

5. Repair Cost Models

In this section, some interesting repair cost models will be analyzed and the optimal repair policy will be characterized.

5.1. Linear Repair Cost Model 1

Suppose that
\[ C_i(a, b) = \begin{cases} K_i(b) + s_i(b)(a-b) & \text{if } a > b \\ 0 & \text{otherwise} \end{cases} \] (23)

where $s_i(b) \geq 0$ is the marginal cost and $K_i(b) \geq 0$ is the fixed cost of repairing the system to deterioration level $b$ during phase $i.$

Lemma 4. If $s_i$ is increasing on $F\setminus\{M\}$ for all $i \in E,$ then Assumption 1 holds.

Proof. Choose arbitrary $b < c < a.$ Then,
\[
C_i(a, c) + C_i(c, b) - C_i(a, b) = K_i(c) + s_i(c)(a-c) + K_i(b) + s_i(b)(c-b) - K_i(b) - s_i(b)(a-b) \\
= K_i(c) + s_i(c)(a-c) + s_i(b)(c-b-a+b) \\
= K_i(c) + (a-c)(s_i(c) - s_i(b)) \\
\geq 0.
\]
Lemma 5. If $s_i$ is decreasing on $F \setminus \{M\}$ for all $i \in E$, then Assumption 3 holds.

Proof. It suffices to show that $\nabla C_i(a, b) \leq \nabla C_i(a + 1, b)$ for arbitrary $a > b$. Then,

$$\nabla C_i(a + 1, b) - \nabla C_i(a, b) = C_i(a + 1, b - 1) - C_i(a + 1, b) - C_i(a, b - 1) + C_i(a, b)$$

$$= s_i(b - 1)(a + 1 - b + 1 - a + b - 1) + s_i(b)(a - b - a - 1 + b)$$

$$= s_i(b - 1) - s_i(b) \geq 0$$

since $s_i$ is decreasing. \hfill \square

Assumption 2 may not hold for this cost structure since

$$\nabla C_i(a + 1, a) - \nabla C_i(a, a) = K_i(a - 1) + 2s_i(a - 1) - K_i(a) - s_i(a) - K_i(a) - s_i(a - 1)$$

$$= s_i(a - 1) - s_i(a) - K_i(a)$$

(24)

and we do not have any information about the sign of the last term.

Lemma 6. If

$$K_i(a) \leq s_i(a - 1) - s_i(a)$$

for all $a \in F \setminus \{M\}$, then Assumption 2 holds.

Proof. Since $K_i(a) \geq 0$, Assumption 3 holds. It suffices to show that $\nabla C_i(a + 1, a) - \nabla C_i(a, a) \geq 0$. Using (24) and the main hypothesis,

$$\nabla C_i(a + 1, a) - \nabla C_i(a, a) = s_i(a - 1) - s_i(a) - K_i(a) \geq 0.$$ 

\hfill \square

Note that if $s_i$ is constant and $K_i = 0$, then both Assumption 1 and Assumption 2 hold. The previous results characterize the optimal repair policy through Theorem 5, Theorem 6, Theorem 7, Theorem 8, and Corollary 4.

Note that all of the results in this section hold if

$$C_i(a, b) = \begin{cases} K_i(a) + s_i(b)(a - b) & \text{if } a > b \\ 0 & \text{otherwise} \end{cases}$$

(25)

5.2. Linear Repair Cost Model 2

Suppose that

$$C_i(a, b) = \begin{cases} K_i(a) + s_i(a)(a - b) & \text{if } a > b \\ 0 & \text{otherwise} \end{cases}$$

(26)

where $s_i(a) \geq 0$ is the marginal cost and $K_i(a) \geq 0$ is the fixed cost of repairing the system with deterioration level $b$ during phase $i$.

Lemma 7. If $s_i$ is decreasing on $F$ for all $i \in E$, then Assumption 1 holds.
Proof. Choose arbitrary \( b < c < a \). Then,

\[
C_i(a, c) + C_i(c, b) - C_i(a, b) = K_i(a) + s_i(a)(a - c) + K_i(c) + s_i(c)(c - b) - K_i(a) - s_i(a)(a - b) \\
= K_i(c) + s_i(c)(c - b) + s_i(a)(a - c - a + b) \\
= K_i(c) + (c - b)(s_i(c) - s_i(a)) \\
\geq 0.
\]

\[\Box\]

Lemma 8. If \( s_i \) is increasing on \( F \) for all \( i \in E \), then Assumption 3 holds.

Proof. It suffices to show that \( \nabla C_i(a, b) \leq \nabla C_i(a + 1, b) \) for arbitrary \( a > b \). Then,

\[
\nabla C_i(a + 1, b) - \nabla C_i(a, b) = C_i(a + 1, b - 1) - C_i(a + 1, b) - C_i(a, b - 1) + C_i(a, b) \\
= s_i(a + 1)(a + 1 - b + 1 - a + b - 1) + s_i(a)(a - b - a - 1 + b) \\
= s_i(a + 1) - s_i(a) \geq 0
\]

since \( s_i \) is increasing.

\[\Box\]

Assumption 2 may not hold for this cost structure since

\[
\nabla C_i(a + 1, a) - \nabla C_i(a, a) = K_i(a + 1) + 2s_i(a + 1) - K_i(a + 1) \\
- s_i(a + 1) - K_i(a) - s_i(a) \\
= s_i(a + 1) - s_i(a) - K_i(a)
\]

and we do not have any information about the sign of the last term.

Lemma 9. If

\[
K_i(a) \leq s_i(a + 1) - s_i(a)
\]

for all \( a \in F \setminus \{M\} \), then Assumption 2 holds.

Proof. Since \( K_i(a) \geq 0 \), Assumption 3 holds. It suffices to show that \( \nabla C_i(a + 1, a) - \nabla C_i(a, a) \geq 0 \). Using (27) and the main hypothesis,

\[
\nabla C_i(a + 1, a) - \nabla C_i(a, a) = s_i(a + 1) - s_i(a) - K_i(a) \geq 0.
\]

\[\Box\]

One can easily see that if \( s_i \) is constant and \( K_i = 0 \), then both Assumption 1 and Assumption 2 hold. The previous results characterize the optimal repair policy through Theorem 5, Theorem 6, Theorem 7, Theorem 8, and Corollary 4.

Note that all of the results in this section hold if

\[
C_i(a, b) = \begin{cases} 
K_i(b) + s_i(a)(a - b) & \text{if } a > b \\
0 & \text{otherwise}
\end{cases}
\]

(28)
5.3. Sell-Purchase Model 1

In this model, if the decision maker gives a decision to repair, then the old device is sold and a better devise is purchased. Let \( s_i(a) \) and \( c_i(a) \) be the salvage value and the purchase cost of a system with deterioration level \( a \) during phase \( i \) respectively. It is assumed that \( s_i \) and \( c_i \) are decreasing functions such that \( c_i \geq s_i \) for all \( i \in E \). Then,

\[
C_i(a, b) = \begin{cases} 
  c_i(b) - s_i(a) & \text{if } b < a \\
  0 & \text{if } b = a.
\end{cases}
\]  

(29)

**Proposition 1.** If \( s_i \) and \( c_i \) are decreasing functions such that \( c_i \geq s_i \), then Assumption 1 and Assumption 3 hold.

**Proof.** Choose arbitrary \( a, b, k \) such that \( b \leq k \leq a \). Then,

\[
C_i(a, k) + C_i(k, b) - C_i(a, b) = c_i(k) - s_i(a) + c_i(b) - s_i(k) - c_i(b) + s_i(a)
\]

\[= c_i(k) - s_i(k) \geq 0 \]

and this implies Assumption 1. Choose arbitrary \( a, b \) such that \( a > b \). Then,

\[
\nabla C_i(a + 1, b) - \nabla C_i(a, b) = C_i(a + 1, b - 1) - C_i(a + 1, b) - C_i(a, b - 1) + C_i(a, b)
\]

\[= c_i(b - 1) - s_i(a + 1) - c_i(b) + s_i(a + 1)
\]

\[= s_i(a) - c_i(a) \leq 0. \]

and this implies Assumption 3.

\[\square\]

It is easy to see that Assumption 2 does not have to hold for this cost structure since

\[
\nabla C_i(a + 1, a) - \nabla C_i(a, a) = C_i(a + 1, a - 1) - C_i(a + 1, a) - C_i(a, a - 1)
\]

\[= c_i(a - 1) - s_i(a + 1) - c_i(a) + s_i(a + 1) - c_i(a - 1) + s_i(a)
\]

\[= s_i(a) - c_i(a) \leq 0. \]

**Theorem 9.** If \( r_i(a) < a \) and \( r_i(a + 1) < a + 1 \), then \( r_i(a + 1) = r_i(a) \) for all \( a \in F \setminus \{M\} \).

**Proof.** Since \( r_i(a) < a \), we have

\[
v(i, a) = \min \left\{ \min_{b \leq a - 1} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \} - s_i(a) , c(i, a) + \Gamma v(i, a) \right\}
\]

\[= \min_{b \leq a - 1} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \} - s_i(a) .
\]

Suppose that \( r_i(a) = k < a \) and, hence,

\[
\min_{b \leq a - 1} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \} = c_i(k) + c(i, k) + \Gamma v(i, k).
\]
Figure 3: A typical optimal repair policy for the sell-purchase model 1.

Since \( r_i(a + 1) < a + 1 \),

\[
V(i, a + 1) = \min \left\{ \min_{b \leq a} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \} - s_i(a + 1), c(i, a + 1) + \Gamma v(i, a + 1) \right\}
\]

= \min_{b \leq a} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \} - s_i(a + 1).

If \( r_i(a + 1) \neq a \), then

\[
\min_{b \leq a-1} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \} \leq c_i(a) + c(i, a) + \Gamma v(i, a)
\]

and hence \( r_i(a + 1) = k \). Therefore, if \( r_i(a + 1) \neq k \), then \( r_i(a + 1) = a \). However, this leads to a contradiction since Assumption 1 holds for this cost structure using Proposition 1 and, hence, \( r_i(a) = a \) by Theorem 5.

Theorem 9 implies that the optimal repair policy for a given phase \( i \) must be as depicted by Figure 3 under this special cost structure. Observe that the optimal policy does not have to be increasing. For levels \( x, x + 1, \) and \( x + 2 \), the optimal policy is to repair to level \( x \). For levels \( y, y + 2, y + 3, \) and \( y + 4 \), the optimal policy is to repair to level \( y \) while the optimal decision is do nothing in \( y + 1 \). Also, note that the optimal policy is do nothing in \( z, z + 1, \) and \( z + 2 \).

5.4. Sell-Purchase Model 2

This model is a special case of the previous model. The only difference between them is that in this model, selling price and purchase price of the system are equal for the same deterioration levels. Then,

\[
C_i(a, b) = c_i(b) - c_i(a)
\]

where \( c_i \) is a decreasing nonnegative function on \( F \) which is selling and purchase price of the system. Özekici and Günlik (1992) shows that Assumption 1, Assumption 2 and Assumption 3 all hold for this cost structure.

**Theorem 10.** If \( r_i(a) < a \), then \( r_i(a) = r_i(a - 1) \) for all \( a \in F \setminus \{M\} \).
Proof. Although this is an immediate corollary of Theorem 8, we provide another proof. Suppose that \( r_i(a - 1) = k \) and \( r_i(a) < a \). We need to show that \( r_i(a) = k \). Since \( C_i(a, b) = c_i(b) - c_i(a) \), we have

\[
v(i, a - 1) = \min_{b \leq a - 1} \{ c_i(b) - c_i(a) + c(i, b) + \Gamma v(i, b) \} = \min_{b \leq a - 1} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \} - c_i(a).
\]

Then, since \( r_i(a - 1) = k \)

\[
\min_{b \leq a - 1} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \} = c_i(k) + c(i, k) + \Gamma v(i, k).
\]  (30)

We know that \( r_i(a) < a \) and, hence,

\[
v(i, a) = \min_{b \leq a - 1} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \} - c_i(a).
\]

Using (30), \( r_i(a) = k \) trivially.

Since Assumption 1 and Assumption 2 hold for this cost structure, the optimal repair policy under this cost structure has the form described in Figure 2.

5.5. A Purchase Model

This cost model is similar to the previous one, but now the salvage value of the old system is zero. Therefore, the cost of a repair is equal to the purchase cost of the better system, namely \( C_i(a, b) = c_i(b) \) if \( b < a \) and \( C_i(a, a) = 0 \) for every phase \( i \in E \). Özekici and Günlik (1992) shows that Assumption 1, and Assumption 3 hold, but Assumption 2 does not hold for this cost structure.

Figure 4: A typical optimal replacement policy for a system performing a mission with two phases.

\[ \begin{align*}
\text{Theorem 11.} & \text{ If } c \text{ is a nonnegative decreasing function on } F \setminus \{M\}, \text{ then there exists } k_i \in F \text{ and } l_i \in F \setminus \{M\} \text{ such that } r_i(a) = a \text{ for all } a < k_i \text{ and } r_i(a) = l_i \text{ for all } a \in \{k_i, k_i + 1, \ldots, M - 1\} \\
& \text{for every } i \in E.
\end{align*} \]

Proof. Choose arbitrary phase \( i \in E \). Let \( k_i \) be the first deterioration level at which the decision maker decides to repair the system, i.e., \( k_i = \inf \{a ; r_i(a) < a\} \). Then, trivially if \( a < k_i \), then
Choose arbitrary programming models. Then, we need to show that \( r_i(k_i) = l_i \) for all \( a > k_i \). Choose arbitrary \( a > k_i \). We have

\[
v(i, k_i) = \min \left\{ \min_{b \leq k_i-1} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \}, c(i, k_i) + \Gamma v(i, k_i) \right\}
\]

and

\[
\min_{b \leq k_i-1} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \} = c_i(l_i) + c(i, l_i) + \Gamma v(i, l_i).
\]

Since, \( v(i, a) \) is increasing in \( a \)

\[
c_i(l_i) + c(i, l_i) + \Gamma v(i, l_i) = v(i, k_i) \leq v(i, a) \leq c(i, a) + \Gamma v(i, a).
\]

Moreover,

\[
v(i, a) = \min \left\{ \min_{b \leq a-1} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \}, c(i, a) + \Gamma v(i, a) \right\}
\]

\[
= \min \left\{ c_i(l_i) + c(i, l_i) + \Gamma v(i, l_i), \min_{k_i \leq b \leq a-1} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \}, c(i, a) + \Gamma v(i, a) \right\}
\]

\[
= \min \left\{ c_i(l_i) + c(i, l_i) + \Gamma v(i, l_i), \min_{k_i \leq b \leq a-1} \{ c_i(b) + c(i, b) + \Gamma v(i, b) \} \right\}
\]

where the last equality follows from (32). Now, choose arbitrary \( b \) such that \( k_i \leq b \leq a - 1 \). Then, since \( v(i, a) \) is increasing in \( a \),

\[
c_i(l_i) + c(i, l_i) + \Gamma v(i, l_i) = v(i, k_i) \leq v(i, b) \leq c(i, b) + \Gamma v(i, b) \leq c_i(b) + c(i, b) + \Gamma v(i, b).
\]

Thus, we have

\[
v(i, a) = c_i(l_i) + c(i, l_i) + \Gamma v(i, l_i)
\]

and, hence, \( r_i(a) = l_i \). \( \square \)

Theorem 11 implies that the optimal repair policy for a system performing a mission with two phases must be as depicted by Figure 4 under this special cost structure. Observe that the optimal policy has a phase dependent control-limit structure specified by the pairs \( (k_1, l_1) \) and \( (k_2, l_2) \). The optimal policy is do nothing for levels \( a < k_i \), and to repair to level \( l_i \) for levels \( a \geq k_i \), for both \( i = 1, 2 \).

6. Appendix

We provide some numerical illustration on our results for both optimal replacement and optimal repair. In particular, we show that if our main assumptions do not hold, then the optimal replacement policy is not necessarily control-limit. In all of our numerical illustrations, the dynamic programming equations are solved by transforming them into appropriate linear programming models.
6.1. Numerical Illustrations on Optimal Replacement

We will show by some counter-examples that the main assumptions on our model in Section 3 are really necessary to guarantee the optimality of a policy with a control-limit structure. As mentioned earlier, we assume that \( p_i \leq f_i \) for all \( i \in E \), \( p_i \) does not depend on the deterioration level of the system and the deterioration process satisfies some monotonicity conditions.

In our examples, we take \( M = 6 \) and the system performs a mission with three phases so that \( E = \{1, 2, 3\} \) and \( F = \{0, 1, \cdots, 6\} \). The transition probability matrix and the transition rates of the mission process are

\[
Q = \begin{bmatrix}
0 & 0.3 & 0.7 \\
0.2 & 0 & 0.8 \\
0.5 & 0.5 & 0
\end{bmatrix}
\quad \text{and} \quad
\mu = \begin{bmatrix}
8 & 1 & 4
\end{bmatrix}.
\]

The transition probability matrices of the deterioration process for each phase are

\[
P_1 = \begin{bmatrix}
0 & 0.1 & 0.2 & 0.2 & 0.3 & 0.1 & 0.1 \\
0 & 0 & 0.1 & 0.15 & 0.2 & 0.25 & 0.3 \\
0 & 0 & 0 & 0.2 & 0.23 & 0.22 & 0.35 \\
0 & 0 & 0 & 0 & 0.3 & 0.3 & 0.4 \\
0 & 0 & 0 & 0 & 0.5 & 0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
P_2 = \begin{bmatrix}
0 & 0.1 & 0.2 & 0.2 & 0.2 & 0.2 & 0.1 \frac{1}{2} \\
0 & 0 & 0.05 & 0.13 & 0.22 & 0.25 & 0.35 \frac{1}{2} \\
0 & 0 & 0 & 0.17 & 0.22 & 0.23 & 0.38 \frac{1}{2} \\
0 & 0 & 0 & 0 & 0.24 & 0.32 & 0.44 \frac{1}{2} \\
0 & 0 & 0 & 0 \frac{1}{2} \\
0 & 0 & 0 & 0 \frac{1}{2} \\
0 & 0 & 0 & 0 \frac{1}{2}
\end{bmatrix},
\]

\[
P_3 = \begin{bmatrix}
0 & 0.1 & 0.1 & 0.1 & 0.15 & 0.2 & 0.35 \frac{1}{2} \\
0 & 0 & 0.05 & 0.08 & 0.22 & 0.25 & 0.4 \frac{1}{2} \\
0 & 0 & 0 & 0.24 & 0.26 & 0.45 \frac{1}{2} \\
0 & 0 & 0 & 0 & 0.45 & 0.55 \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 \frac{1}{2} \\
0 & 0 & 0 & 0 \frac{1}{2} \\
0 & 0 & 0 & 0 \frac{1}{2}
\end{bmatrix},
\]

and the related transition rates are

\[
\lambda_1 = [4 \ 5 \ 5.5 \ 9 \ 80,000 \ 90,000], \quad \lambda_2 = [2 \ 3 \ 5 \ 6 \ 620,000 \ 650,000],
\]

\[
\lambda_3 = [4 \ 5 \ 6 \ 7 \ 80,000 \ 10,0000].
\]

The maintenance and failure costs are \( p_1 = 200, p_2 = 10, p_3 = 30, f_1 = 300, f_2 = 50 \) and \( f_3 = 80 \). The discount rate is \( \alpha = 0.8 \) and all state occupancy costs are 0. Unless otherwise specified, these parameters will be used in all of our examples. This is our base model and we will produce the counter-examples by changing some parameters in this base model. In all of the tabular representations through this paper, if it is not clear from the context, we suppose that the rows correspond to the phases of the mission while the columns represent deterioration levels.

The optimal replacement policy and the optimal costs are

\[
s^* = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\]
and
\[
v^* = \begin{bmatrix} 211.790 & 251.023 & 265.112 & 311.449 & 411.790 & 411.790 & 511.790 \\ 122.279 & 132.279 & 132.279 & 132.279 & 132.279 & 132.279 & 172.279 \\ 177.180 & 193.980 & 201.191 & 207.180 & 207.180 & 207.180 & 257.180 \end{bmatrix}
\]
in the base case. It is clear that this is a control-limit policy and the critical thresholds are \(a_1^* = 4, a_2^* = 1, a_3^* = 3\).

Example 1. In this example, we show that if \(f_i < p_i\) for some \(i \in E\), then the optimal replacement policy does not have to be control-limit and the value function does not have to be increasing in the deterioration level of the system. Suppose now that \(p_1 = 200, p_2 = 10, p_3 = 1, f_1 = 210, f_2 = 11\) and \(f_3 = 0.8 < p_3 = 1\). The discount rate is \(\alpha = 0.99\). Using these parameters, the optimal replacement policy and optimal costs are

\[
r^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\]

and
\[
v^* = \begin{bmatrix} 104.522 & 129.644 & 139.025 & 180.017 & 304.522 & 304.522 & 314.522 \\ 42.876 & 46.638 & 48.754 & 50.627 & 52.876 & 52.876 & 53.876 \\ 59.768 & 60.768 & 60.768 & 60.768 & 60.575 & 60.572 & 60.658 \end{bmatrix}.
\]

It is obvious that the optimal cost function is not increasing and the optimal policy is not control-limit for phase 3. This is due to the fact that the preventive replacement cost is higher than the failure replacement cost in this phase.

Example 2. In this example, we show that if \(p_i\) depends on the deterioration level of the system, then the optimal policy does not have to be control-limit. Suppose that

\[
Q = \begin{bmatrix} 0 & 0.4 & 0.6 \\ 0.3 & 0 & 0.7 \\ 0.5 & 0.5 & 0 \end{bmatrix}, \quad \mu = [5 \ 8 \ 4],
\]

\[
\lambda_1 = [1 \ 2 \ 3 \ 3.1 \ 3.2 \ 3.3], \quad \lambda_2 = [2 \ 3 \ 5 \ 80 \ 1000 \ 1200],
\]

\[
\lambda_3 = [1 \ 1.5 \ 2.5 \ 4 \ 4.5 \ 5],
\]

\[
p = \begin{bmatrix} 15 & 20 & 25 & 30 & 35 & 40 \\ 15 & 20 & 25 & 30 & 35 & 40 \\ 15 & 20 & 25 & 30 & 35 & 40 \end{bmatrix}, \quad f = [76 \ 45 \ 41]
\]

and \(\alpha = 0.75\). Then, the optimal replacement policy and optimal costs are

\[
r^* = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

and
\[
v^* = \begin{bmatrix} 51.709 & 67.839 & 76.709 & 81.709 & 86.487 & 91.709 & 127.709 \\ 54.330 & 68.141 & 76.872 & 84.330 & 89.330 & 94.330 & 99.330 \\ 51.340 & 64.182 & 72.452 & 78.269 & 81.443 & 85.516 & 92.340 \end{bmatrix}.
\]
It is clear that although the optimal cost function $v(i,a)$ is increasing in $a$ for all $i$, the optimal policy is not control-limit for phase 1. This is due to the fact that the expected future cost after a preventive replacement increases in the deterioration level of the system. Hence, it can be higher than the expected future cost after a “do nothing” decision at a deterioration level because of discounting.

**Example 3.** We now show that if deterioration process of the system for a given phase does not satisfy (1) and (2), then the optimal policy does not have to be control-limit. Suppose that $p_1 = 20, \ p_2 = 10, \ p_3 = 30, \ f_1 = 300, \ f_2 = 150, \ f_3 = 180,$

$$\lambda_1 = [4 \ 5 \ 150 \ 180 \ 3 \ 5], \ \lambda_2 = [2 \ 3 \ 200 \ 220 \ 3 \ 5],$$

$$\lambda_3 = [4 \ 5 \ 300 \ 350 \ 3 \ 5],$$

and

$$P_1 = P_2 = P_3 = \begin{bmatrix} 0 & 0.1 & 0.1 & 0.15 & 0.2 & 0.35 \\ 0 & 0 & 0.4 & 0.08 & 0.1 & 0.1 & 0.32 \\ 0 & 0 & 0 & 0.55 & 0.13 & 0.12 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$  

Clearly, conditions (1) and (2) are violated. Then, the optimal policy and optimal costs are

$$r^* = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and

$$v^* = \begin{bmatrix} 270.753 & 283.442 & 290.753 & 281.886 & 241.782 & 290.753 & 570.753 \\ 208.036 & 218.036 & 218.036 & 218.036 & 203.278 & 218.036 & 358.036 \\ 258.427 & 270.443 & 288.427 & 266.345 & 236.590 & 288.427 & 438.427 \end{bmatrix}.$$  

It is clear that the optimal policy is not control-limit and the optimal costs are not increasing in the deterioration level of the system for each phase. This result follows from the fact that the system may tend to be more reliable as the deterioration level of the system increases when its deterioration process does not satisfy the increasing failure rate conditions (1) and (2).

6.2. Numerical Illustrations on Optimal Repair

The following example shows that if neither Assumption 2 nor Assumption 3 holds, then $r_i$ does not have to be increasing in the deterioration level of the system.

**Example 4.** Consider the base problem in Section 6.1. Suppose that the transition rates of the deterioration process are

$$\lambda_1 = [4 \ 5 \ 5.5 \ 9 \ 9200 \ 9500], \ \lambda_2 = [2 \ 3 \ 5 \ 6 \ 7000 \ 8000],$$

$$\lambda_3 = [4 \ 5 \ 6 \ 7 \ 8000 \ 9000],$$
the cost matrix is

\[
C_i = \begin{bmatrix}
0 & - & - & - & - & - & - \\
700 & 0 & - & - & - & - & - \\
800 & 300 & 0 & - & - & - & - \\
900 & 330 & 100 & 0 & - & - & - \\
1000 & 500 & 120 & 100 & 0 & - & - \\
2000 & 1140 & 1130 & 1100 & 1100 & 0 & - \\
2500 & 1900 & 1800 & 1600 & 1400 & 1200 & 0 \\
\end{bmatrix}
\]

for all \( i \in E \) and \( \alpha = 0.80 \). Note that \( \nabla C_i(1, 1) = 700, \nabla C_i(2, 1) = 500, \nabla C_i(3, 1) = 570, \nabla C_i(4, 1) = 500 \) and, hence, both Assumption 2 and Assumption 3 do not hold. For these parameters, the optimal repair policy is

\[
r = \begin{bmatrix}
0 & 1 & 2 & 2 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 2 & 1 & 2 & 1 & 0 \\
\end{bmatrix}.
\]

It is clear that \( r_i \) is not monotone.

At first glance, intuition may say that if \( C_i(a, b) \) is increasing in \( a \) and decreasing in \( b \), and if Assumption 1 and Assumption 2 hold, then \( r_i(a) < a \) implies that \( r_i(a+1) < a+1 \). However, the following example shows that this is not always true.

**Example 5.** Consider Example 4. Suppose that the transition rates of the deterioration process are

\[
\lambda_1 = [4 \ 5 \ 5.5 \ 9 \ 9.2 \ 9.5], \lambda_2 = [2 \ 3 \ 5 \ 6 \ 7 \ 8], \lambda_3 = [4 \ 5 \ 6 \ 7 \ 8 \ 9],
\]

the cost matrix is

\[
C_i = \begin{bmatrix}
0 & - & - & - & - & - & - \\
10 & 0 & - & - & - & - & - \\
30 & 20 & 0 & - & - & - & - \\
50 & 40 & 20 & 0 & - & - & - \\
1000 & 990 & 970 & 950 & 0 & - & - \\
2500 & 2490 & 2470 & 2450 & 1500 & 500 & 0 \\
\end{bmatrix}
\]

for all \( i \in E \) and \( \alpha = 0.80 \). It is clear that \( C_i(a, b) \) is increasing in \( a \) and decreasing in \( b \), and Assumption 1 and Assumption 2 hold. For these parameters, the optimal repair policy is

\[
r = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & 0 \\
\end{bmatrix}.
\]

Note that \( r_1(4) = 0 < 4 \) and \( r_1(5) = 5 \). This result shows that \( r_i(a) < a \) does not imply that \( r_i(a+1) < a+1 \) even when Assumption 1 and Assumption 2 hold. This result is also very intuitive since a preventive maintenance decision may not be viable when system failure is very likely in the near future.
References


